A Note on Optimal Algorithms for Fixed Points

S. Shellman, K. Sikorski
School of Computing, University of Utah
Salt Lake City, UT 84112

22nd February 2010

Abstract

We present a constructive lemma that we believe will make possible the design of nearly optimal \( O(d \log \frac{1}{\varepsilon}) \) cost algorithms for computing \( \varepsilon \)-residual approximations to the fixed points of \( d \)-dimensional nonexpansive mappings with respect to the infinity norm. This lemma is a generalization of a two-dimensional result that we proved in [1].

1 Introduction

In [1, 2] we presented two-dimensional optimal complexity algorithms for computing residual \( \varepsilon \)-approximations to the fixed points of non-expansive mappings with respect to the infinity norm. These algorithms are based on bisection-envelope constructions and are derived from Theorem 3.1 of [1]. This theorem makes possible construction of a sequence of rectangles that contain fixed points and converge to the residual \( \varepsilon \)-approximation of some fixed point. At every iteration of the process the previous rectangle is cut by a factor of at least two, to obtain a new rectangle containing a fixed point.

In this paper we generalize the constructive theorem to an arbitrary number of dimensions \( d \geq 3 \), however, we are unable to utilize this new result in the construction of optimal algorithms.

The main obstacle in such construction is the ability to bound a new set containing fixed points by an "easy-to-construct" convex set of smaller volume and similar topological features to the previous set in this process. We stress that the two-dimensional sets in the optimal algorithm are rotated rectangles. What would be the proper sets in an arbitrary number of dimensions that would bound the non-convex sets resulting from the application of our general \( d \)-dimensional lemma?

2 Problem formulation

Given dimension \( d \geq 2 \), we define \( D = [0,1]^d \) and the class \( F \) of functions, \( f : D \to D \), that are Lipschitz continuous with constant 1 with respect to the
infinity norm, i.e.,
\[ \|f(x) - f(y)\| \leq \|x - y\|, \forall x, y \in D \]

where \( \|\cdot\| = \|\cdot\|_\infty \) henceforth. We seek an algorithm which, for every \( f \in F \), computes a solution \( \hat{x} = \hat{x}(f) \in D \) that satisfies the residual criterion
\[ \|f(\hat{x}) - \hat{x}\| < \epsilon \quad (1) \]

where \( 0 < \epsilon < 0.5 \). (If \( \epsilon \geq 0.5 \) then \( x = (0.5, 0.5) \) satisfies \( [1] \)). The algorithm requires \( n(f) \) function evaluations, where \( n(f) \approx O(d \log \frac{1}{\epsilon}) \). In the case of \( d = 2 \) the algorithm is based on Theorem 3.1 of [1], utilizes bisection of rectangles and envelope constructions, and has cost \( 2 \log_2 \frac{1}{\epsilon} \). Here we present a generalization of this theorem to the case of \( d \geq 3 \). We believe that the general result will provide the basis for construction of a future algorithm having the desired efficiency. So far we have been unable to construct such an algorithm. We stress that computing \( x_\epsilon, \|x_\epsilon - \alpha\| < \epsilon \), an \( \epsilon \)-absolute approximation to the fixed point \( \alpha \), in the class of expanding functions is of infinite complexity in the worst case [3].

3 Definitions

For a given \( f \in F \) and \( i = 1, \ldots, d \) we define the fixed point sets \( F_i \) such that for each \( i \),
\[ F_i(f) = \left\{ x \in D : f_i(x) = x_i \right\}. \]

We define \( F(f) = \bigcap_{i=1}^{d} F_i(f) \), the nonempty set of all fixed points of \( f \). For all \( x \in \mathbb{R}^d, i = 1, \ldots, d, \) and \( s \in \{-1, 1\} \) we define the “open-ended” pyramid sets
\[ A_i^s(x) = \left\{ y \in \mathbb{R}^d : \|y - x\| = s(y_i - x_i) \right\}. \]

For all \( x \in \mathbb{R}^d, i = 1, \ldots, d, s \in \{-1, 1\}, \) and \( c > 0 \), we also define the “flat-top” pyramid set
\[ Q_i^s(x, c) = \bigcup \left\{ A_i^s(y) : y \in \mathbb{R}^d, \|y - x\| < c \right\}. \]

4 Constructive Lemma

In this section we prove our constructive lemma. It is a generalization of Theorem 3.1 of [1] to an arbitrary number of dimensions \( d \geq 3 \).

Lemma 4.1

For any \( f \in F, i = 1, \ldots, d \), we let \( x \in D \) be such that \( f_i(x) \neq x_i \). Then the following holds:
(i) If \( f_i(x) > x_i \) then \( Q_i^{-1}(x, (f_i(x) - x_i)/2) \cap D \cap F_i(f) = \emptyset \).
Proof. To show (i) we take any \( y \) such that \( \|y - x\| < (f_i(x) - x_i)/2 \), and \( z \in A_i^{-1}(y) \cap D \). Then

\[
|f_i(z) - f_i(y)| \leq \|f(z) - f(y)\| \leq \|z - y\| = y_i - z_i
\]

and

\[
f_i(y) - y_i = f_i(x) - (f_i(x) - f_i(y)) - x_i - (y_i - x_i) \geq f_i(x) - x_i - 2 \|y - x\|
\]

\[
> f_i(x) - x_i - (f_i(x) - x_i) = 0,
\]

which implies

\[
f_i(z) = f_i(y) + (f_i(z) - f_i(y)) > y_i - (y_i - z_i) = z_i.
\]

To show (ii) we take any \( y \) such that \( \|y - x\| < (x_i - f(x_i))/2 \), and \( z \in A_i^{-1}(y) \cap D \). Then

\[
|f_i(z) - f_i(y)| \leq \|f(z) - f(y)\| \leq \|z - y\| = z_i - y_i
\]

and

\[
f_i(y) - y_i = f_i(x) + (f_i(y) - f_i(x)) - x_i + (x_i - y_i) \leq f_i(x) - x_i + 2 \|y - x\|
\]

\[
< f_i(x) - x_i + (x_i - f_i(x)) = 0,
\]

which implies

\[
f_i(z) = f_i(y) + (f_i(z) - f_i(y)) < y_i + (z_i - y_i) = z_i.
\]

\[\square\]

Comments

The above Lemma 4.1 states that after evaluating \( f \) at \( x \) we can remove from the original domain \( D \) the “flat-top” pyramid sets \( Q^*_i(x, c_i) \) for all \( i \) such that \( c_i = |f(x_i) - x_i|/2 \) are not zero, since they do not contain fixed points of \( f_i \), implying that they do not contain any fixed point of \( f \) as well. If this happens for all \( i = 1, \ldots, d \) then we can reduce the volume of the set containing fixed points by a factor of at least two.

Open problems

The main obstacle in constructing a recursive algorithm (for \( d \geq 3 \)) based on Lemma 4.1 is our apparent inability to construct a sequence of sets \( S_j \) that each contain a fixed point, are topologically “similar”, decrease in volume, and are easy to represent, and then evaluating \( f \) at the “centers” of \( S_j \). Also, it needs to be decided which sets can be removed from \( S_j \) in the case where \( f_i(x) - x_i = 0 \), i.e., when the current evaluation point \( x \) is a fixed point of some components of \( f \).

We believe that by solving those problems we can obtain an optimal \( O(d \log \frac{1}{\epsilon}) \) cost algorithm for finding \( \epsilon \)-residual solutions to the fixed points of functions in our class. We hope to address these issues in a future paper.
References


